THE NATURE OF ALGEBRAIC ABILITIES*

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THE PRESENT CONCEPTION OF ALGEBRA AS A SCHOOL SUBJECT

During the generation from 1880 to 1910 which witnessed the popularization of high schools in America, algebra[1] became fixed as a required first year study, and with a content which I shall call for convenience the "older" content, or the "older" algebra. The "older" algebra sought to create and improve the following abilities: to read, write, add, subtract, multiply, divide, and to handle ratios, proportions, powers and roots with negative numbers and literal expressions, to "solve" equations and sets of equations, linear and quadratic, and to use these techniques in finding the answer to problems. These abilities were interpreted very broadly in certain respects and very narrowly in others. If anybody had asked Wentworth, for example, what negative numbers and literal expressions the pupil should be able to add, he would probably have answered, "Any"; and the pupils did indeed add an enormous variety, including many which were never experienced anywhere in the world outside of the school course in algebra.[2] On the other hand, decimals were very rarely used, and angles were almost never added, in spite of the definite need for that ability in the geometry of the following year.

The actual content with which these abilities were trained was determined largely by two forces. The first was faith in indis-

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1 Throughout these articles "Algebra" will be used for what is commonly called in this country "Elementary Algebra."

2 For example:

1. Add: $4x^4y^5z^6 - 3x^3y^4z^5 + 17x^2y^3z^4 - 8xy^2z^3$; $14x^2y^3z^4 + 4xy^2z^3 + 5x^3y^4z^5 - 3x^4y^5z^6$; $-4x^4y^5z^6 - 2x^3y^4z^5 + 4xy^2z^3 + 19x^2y^3z^4$; $2x^3y^4z^5 + 5xy^2z^3 - 7x^4y^5z^6 + 9x^2y^3z^4$; $-12xy^2z^3 + 4x^4y^5z^6 - 15x^2y^3z^4 - x^3y^4z^5$; $3x^4y^5z^6 + 41x^2y^3z^4 - x^3y^4z^5 + 7xy^2z^3$.

Pg. 51, ex. 11; Wentworth: Elementary Algebra. Edition 1906; (reprinted from older editions).

criminate thought and practice—the resulting tendency being to have the pupil add, subtract, multiply and divide, anything that could be added, subtracted, multiplied or divided; and to have him solve any problem that the teacher could devise. The second was the inertia of custom, the resulting tendencies being, among others, to make algebra parallel arithmetic, to continue puzzle problems, to use applications conceived before or apart from the growth of quantitative work in the physical sciences, and to be unappreciative of graphic methods of presenting facts and relations.

The faith in indiscriminate reasoning and drill was one aspect of the faith in general mental discipline, the value of mathematical thought for thought's sake and computation for computation's sake being itself so great that what you thought about and what you computed with were relatively unimportant.

The paralleling of arithmetic was perhaps most noticeable in the order of topics, and in the almost monomaniac devotion to problems with one particular set of quantities and conditions so that there was some one number as the "answer." There was no reason why $a \times a = a^2$ and $a \times a^2 = a^3$ should not have been taught before $a + a = 2a$, but to do so probably never even occurred to the generation of teachers in question. That a general relation as an answer was a much more important matter than the number of miles a particular boat went, or the number of dollars a particular boy had, and more suitable as a test of algebraic achievement—this again hardly entered their minds.

This older algebra survives in whole or in part in some courses of study, instruments of instruction, and examination procedures. As the accepted view of leaders in the teaching of mathematics and in general educational theory, it is, however, now a thing of the past. I shall use the word "algebra" from now on to refer to the algebra which these leaders recommend as content for teaching in grade nine (sometimes grades nine and ten), or as a part of the mathematics of grades eight, nine and ten.

These leaders are not, of course, in exact agreement concerning details of content and degrees of emphasis, but, approximately, they would subtract from and add to the "older" algebra as follows:

They would omit such computations as occur never or very seldom outside of the older algebra. Addition, subtraction, multiplication and division with very long polynomials, special products except $(a+b)^2$ $(a-b)^2$ $(a+b)(a-b)$ $(ax+b)(cx+d)$, the corresponding factorizations, fractions with polynomials in the denominator more intricate than $a(b+c)$, elaborate simplifications involving nests of brackets, compound and complex fractions, and rationalizations other than of $\sqrt{a}$, $\sqrt{a}+\sqrt{b}$, $\sqrt{a}-\sqrt{b}$, L. C. M.'s and H. C. F.'s except such as are obtainable by inspection—all these are taboo except in so far as some emphatic need of the other sciences or of mathematics itself requires the technique in question. Clumsy traditions in ratio and proportion (such as the use of "means," "extremes," "antecedent" and "consequent") are eliminated. Bogus and fantastic problems are forbidden wherever a genuine and real problem is available that illustrates or applies the principles as well. The actual uses of algebra in mathematics, science, business, and industry are canvassed and merit is attached to those abilities which are of service there. The mere fact that an operation, *e. g.*,

$$(2a^4+3a^3b^2c-7bc^2d-8d^2)(a-3b^2)$$

can be performed is not a sufficient reason for asking school pupils to perform it. The mere fact that a problem can be framed is not a proof that pupils will profit from solving it.

Thus from one-fourth to one-half of the time spent on the older algebra is saved. This is used to establish and improve the following abilities:

To understand formulae, to "evaluate" a formula by substituting numbers and quantities for some of its symbols, to rearrange a formula to express a different relation,* to compute with line segments, angles, important ratios, and decimal coefficients, to understand simple graphs, to construct such graphs from tables of related values, and to understand the Cartesian coordinates so as to use them in showing simple relations of y to x graphically.

The discussion of Nunn (1914) and Rugg and Clark (1917), the reports of the Central Association of Science and Mathe-

* i. e., "Changing the subject" of the formula, or "solving" for one of the variables without substituting particular values.

matics Teachers (1919) and the new requirements under consideration by various students of school and college examinations would, if combined into an average consensus, tally rather closely with the foregoing statement.

Whereas the older algebra, giving in the main an indiscriminate acquaintance with negative and literal numbers and their uses, expected an undefined improvement of the mind, this algebra is selective and expects to improve the mind by extending and refining its powers of analysis, generalization, symbolism, seeing and using relations, and organizing data to fit some purpose or question. It expects to improve these greatly for algebraic analyses, generalizations, symbolisms, and relations and for the organization of a set of quantitative facts and relations as an equation or set of equations, and hopes for a profitable amount of transfer to analyses, generalizations, symbolisms, relational thinking and organizations outside of algebra. It expects further to give better special preparation to see the more direct needs for algebra in life at large and to use it to meet them effectively.

This program for algebra is fairly clear and comprehensible, as educational programs go. Neveretheless, a hundred teachers and a hundred psychologists and a hundred mathematicians who should try to act on it as stated, would probably do three hundred things, no two of which would be identical.

We need fuller and more exact statements of the nature of algebraic abilities and of the uses of algebra in mathematics, science, business and industry. In particular we need clearer knowledge of what is, and what should be, meant by "ability to understand formulae," "ability with equations," "ability to solve problems," and "ability to understand, make and use graphs." Still more do we need clearer knowledge of what "analysis," "generalization," "symbolism," "thinking with relations," and "organization" mean.

Ability to Understand and Frame Formulae

The ability to understand formulae may mean simply the ability to understand the face value of the symbols involved. Such is the case when a pupil understands that $A = p + prt$ means

the "amount is equal to the principal plus the product of the principal, rate and time," or, being given the formula and also:

Let the case be one of simple interest, and let the interest accrue without fixed reinvestment,
Let $A =$ the amount in dollars,
Let $p =$ the principal in dollars,
Let $r =$ the percent paid per year for the use of the money, and
Let $t =$ the time in years,

he understands that $A = p + prt$ means "Fill in p, r and t and A will be the correct amount."

The ability to understand formulae may, however, mean the ability to understand the face value of the symbols and also to supply such units and make such interpretation of the situation and the result of using the formula as fits the case and insures the right answer. Thus if, in the case above, the pupil was given only $A = p + prt$, and knew when and how to use it he would really understand much more than the formula. Many pupils, for example, who could translate $A = p + prt$ and use it as they had been taught to do habitually would fail with "What would be the amount of 74 pounds at 1% per month after eight years, the interest being paid every 2 years but left uninvested?" They would not know how to use the formula or even perhaps whether to use it.

The extent to which pupils shall be expected to read between the lines of a formula, knowing when it applies and when it does not, and choosing such units that the result will be correct is a matter of dispute in theory and practice.

On the one hand it is argued that such interpretations are a matter of physics or geometry or business practice or the like, not algebra, and also that the mixture of such interpretations with rigorous mathematical thinking, lessens the instructiveness of the latter. Algebraically, for example, it is correct if $A = p + prt$ and $I = \frac{E}{R}$ to conclude that $AI = \frac{pE + prtE}{R}$. That it happens to be nonsense to say that the amount of money times the current equals the principal times the voltage, etc., is not for the learner of algebra to know or care. So the extremists might argue.

On the other hand, it is argued, first, that algebraic technique divorced from its applications to lengths and weights and dollars and years and amperes and volts is a barren game; second, that absolute clearness and rigor in the statement of formulae so that nothing needs to be read between the lines spoils the best feature of a formula, its brevity. Only two principles are needed, the extremists on this side would say. First, "Use formulàe only in ways such as common sense and the facts of the case tell you are reasonable." Second, "Use such units that the answer will be right."

From the point of view of the psychology of the learner either extreme seems tolerable, provided it is operated with consistency and frankness, and provided, in the case of the second plan, too much sacrifice of comprehensibility to brevity is not made. The learner may be taught to insist that every symbol in a formula be defined as a quantity, expressed as a number of such and such units, and to separate sharply his operations with a formula from his choice of which formula. He would then simply refuse to try to operate with most formulae as commonly given. $I = \frac{E}{R}$ would have to be defined as if $I =$ the current in amperes; $E =$ the potential in volts and $R =$ the resistance in ohms,—then $I = \frac{E}{R}$ $A = \frac{1}{2}BH$ for a triangle would have to be extended to:—"Let $B =$ the number of inches in the base of the triangle. Let $H =$ the number of inches in the altitude of the triangle. Let $A =$ the number of square inches in the area of the triangle. Then $A = \frac{1}{2}BH$. If B and H are numbers of feet, A will equal the number of square feet in the area of the triangle," etc., etc. If he chooses the right formula the result of correct computation is ipso facto the right answer.

He may, on the contrary, be taught that most formulae, such as $I = \frac{E}{R}$ or $S = at + \frac{1}{2}gt^2$, are simply hints to guide memory and thought in framing the right choice and arrangement of symbols and numbers, and that he is responsible for that arrangement, and for the interpretation of any results of evaluating or solving it.

The former plan secures abilities easier to learn and requires less skill in the teaching; the latter secures abilities which are perhaps more educative and a better preparation for dealing with formulae as they actually occur in books on science and technology. If so, however, it is because time and thought are spent in the algebra course in learning science and technology, or in solving ambiguities of statement by reasoning out what probably is or should be meant.

The greatest danger in the second plan is in the pupil's framing of formulae. Suppose, for example, that he is told to express in a formula the fact that Profit equals Sales less the Number of Articles Produced times the Production Cost per Article, less Selling Costs plus Overhead, and writes $P = S - NC_p - C_s + O$. Is he to be blamed? His algebra and symbolism are correct. It is only his knowledge of business facts and terms that is at fault. If he writes $P = S - NC_p - C_s - O$, is he to be praised? C_p in actual business may well be a number of cents, not dollars, so that his formula may produce a preposterous answer. Or suppose that he is asked to frame a formula for the number of acres in a rectangular plot, the length and width in feet being given, and writes $A = \frac{lw}{a}$. This is true enough if a is correctly defined "between the lines" as the number of square feet in one acre, but of what use is it? In *framing* formulae it seems best to teach the pupil to demand such rigor and adequacy in the conditions given to him that his task is simply translation into an arrangement of numbers and symbols and to demand the same rigor and adequacy of him.

In *reading* formulae it seems reasonable to train the pupil to a certain extent to read between the lines, to be judicious and consistent in his selection of units, and in other respects to use formulae as suggestions and clues rather than as adequate, unambiguous rules. It will be convenient and probably sometimes necessary for him to do so in his actual contacts with formulae in books and elsewhere.

In either case the pupil may profitably understand that from the moment that he begins to operate with the formula until he completes the operations by reaching the desired result or "an-

swer'' all the symbols are simply numbers. Nothing needs to be labeled as inches, feet, dollars, years, volts, ohms, foot—pounds, or the like during the operations. What the quantities are must be considered before operating in choosing or framing the formula, and after operations are done in order to put the right label or interpretation on the ''answer.'' But for the purposes of operation $\text{Amperes} = \frac{\text{volts}}{\text{ohms}}$ is just like $\text{Ans.} = \frac{\text{Number } a}{\text{Number } b}$ or $x = \frac{a}{b}$

Other things being equal, genuine formulae useful in mathematics, science, industry and business are to be preferred for training in understanding ,evaluating, transforming and framing formulae. Other things, especially convenience, are not always equal. The genuine formulae that are of significance to pupils may be too simple, or too much burdened with long numbers, and there may not be enough of them to give the practice considered necessary. So teachers and textbooks tend to make up formulae of just the desired complexity, involving just the relations with which practice is needed, and with just as little or much numerical difficulty as the occasion demands.

The use of these artificial formulae is not essentially more vicious than the use of multiplications like 465×9817. It is probable that not one pupil in a hundred will ever have to multiply 9817 by 465. But we do not object to such work in moderation in arithmetic because the elementary abilities practiced are all useful; and this is a good way to give them practice. In the same way practice with a formula like $P = M^2N + \frac{M(O-N)}{N}$ may be defensible although the formula has only a very slight probability of occurence outside of school.

Ability with Equations

Ability with equations includes two groups of abilities which are, at least psychologically, very different. The first is to manipulate the equation so as to obtain a numerical value for the literal element, or so as to obtain a value for one of the

literal elements in terms of the others. The equation is "solved." The second is to understand the equation as the expression of a certain relation whereby we can correctly prophecy what value a certain element will have, according to the values which one or more other elements have.

Thus $\frac{Q}{2} = KR + 4$ is "solved" for Q by finding that $Q = 2KR + 8$. $\frac{Q}{2} = KR + 4$ is "understood with respect to the relation between Q and R" by understanding that if K is a constant, Q is in direct proportion to R, that if Q is expressed as ordinate to fit R abscissa value, we have a set of points on a straight line cutting the y axis at $+8$, with a slope depending on what K is, and that every increment added to R produces, other things being equal, an increment of $2K$ in Q. The older algebra neglected the second ability almost entirely, and even yet the first ability is given far more time and attention in most textbooks, courses, and examinations. Yet the second ability seems of equal or greater importance.

There are three cases of "solving." First, the pupil is taught to organize all the data needed to secure the answer to his problem in the form of an equation with x or n or Ans. or ? or an empty space to be filled, to represent his desired result. "Solving" then means the computation needed to get the x or n or Ans. or ? or empty space on one side of $=$ and to get the other side free from the x or n and where desirable, in simplest form. Sometimes two or more equations with x and y or n_1 and n_2 or Part I of Ans. and Part II of Ans. are used. The ability to manage this organization and manipulation of data is useful. The problems of life, when of this sort, almost never lead to quadratic equations. The computations are rarely literal.

Second, the pupil is taught to "solve" a formula or equation already organized as when he derives a formula for finding the radius of a circle from its circumference, from $C = 2\pi r$.

Third, the pupil is taught to solve equations of the type $y = ax + b$ or $y = x^2 + ax + b$ for a and b, being given related pairs of values of x and y, and to discover the two values of x corresponding to a given value of y in equations of the

type, $y = ax^2 + bx + c$. This third sort of "solving" is valuable if a certain mastery of the "understanding of the relation" has been attained. Otherwise it is dangerously near to being an aimless mental gymnastic. The older but still common practice of solving quadratics only for the case where $y = 0$, out of all relation to the general problem, seems indefensible. The only argument in its favor seems to be that x in $ax^2 + bx + c = 0$ is an unknown quantity and that you should therefore find its value regardless of whether knowledge of its value is of any consequence.

The understanding of equations as the expression of relations, goes straight to the heart of all applied mathematics, showing the formula and equation as the story of a rule or law which certain events in nature follow or approximate; it introduces the most important idea of mathematics, that of quantitative dependence or functionality; it is a vital and potent review of the principle that algebra tells what will happen to *any* number under certain conditions; it furnishes a principle of organization for graphics; it furnishes the treble parallelism between certain important relations, certain graphs and certain equations which will arouse respect for algebra.

It may be retorted that the understanding of an equation as a story of a relation or law is too hard and varied an ability for pupils in the ninth or tenth grade to acquire, in comparison with the more mechanical and uniform "solving." We shall see in a later article that it has been made needlessly difficult by unfortunate usage of terms and unwise building up of certain mental habits which get in each other's way and trip each other up; and we shall there show how to reduce these difficulties greatly.